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Some Functional Inequalities and Inclusion Relationships Associated with Certain Families of Integral Operators

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Abstract—By making use of certain familiar integral operators, the authors introduce and investigate several new subclasses of starlike, convex, close-to-convex, and quasi-convex functions Among other results presented here, the authors establish a number of inclusion relationships associated with some of these integral operators. Some of the results established in this paper would provide extensions of those given in earlier works

Keywords—Analytic functions, Univalent functions, Starlike functions, Convex functions, Close-to-convex functions, Quasi-convex functions, Jung-Kim-Srivastava operators, Bernardi operator, Miller-Mocanu lemma, Multiplier transformation, Fractional integral, Fractional derivative

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Suppose also that $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all functions which are *univalent* in $\mathbb{U}$.

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We denote by $\mathcal{S}^*(\alpha)$, $\mathbb{K}(\alpha)$, $\mathcal{C}(\beta,\alpha)$, and $\mathcal{C}^*(\beta,\alpha)$, the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, *starlike of order* α in $\mathbb{U}$, *convex of order* α in $\mathbb{U}$, *close-to-convex of order* β *and type* α in $\mathbb{U}$, and *quasi-convex of order* β *and type* α in $\mathbb{U}$. Thus, by definition, we have (for details, see [1–4])

$$\mathcal{S}^*(\alpha) := \left\{f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (z \in \mathbb{U};\ 0 \leqq \alpha < 1)\right\}, \tag{1.2}$$

$$\mathbb{K}(\alpha) := \left\{f : f \in \mathcal{A} \text{ and } \Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \ (z \in \mathbb{U};\ 0 \leqq \alpha < 1)\right\}, \tag{1.3}$$

$$\mathcal{C}(\beta,\alpha) := \left\{f : f\in\mathcal{A},\ g\in\mathcal{S}^*(\alpha), \text{ and } \Re\left(\frac{zf'(z)}{g(z)}\right) > \beta \ (z\in\mathbb{U};\ 0\leqq\alpha,\ \beta<1)\right\}, \tag{1.4}$$

and

$$\mathcal{C}^*(\beta,\alpha) := \left\{f : f\in\mathcal{A},\ g\in\mathbb{K}(\alpha), \text{ and } \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta \ (z\in\mathbb{U};\ 0\leqq\alpha,\ \beta<1)\right\}. \tag{1.5}$$

It is easily observed from the definitions (1.2)–(1.5) that

$$f(z) \in \mathbb{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \tag{1.6}$$

and

$$f(z) \in \mathcal{C}^*(\beta,\alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}(\beta,\alpha). \tag{1.7}$$

In particular, the class $\mathcal{C}^*(\beta,\alpha)$ of quasi-convex functions of order β and type α in $\mathbb{U}$ was considered recently by Noor [4].

Several interesting families of integral operators, which have been investigated rather extensively in *analytic function theory*, including each of the following integral operators,

$$\begin{aligned} P_\lambda^\mu f(z) &:= \frac{(\lambda+1)^\mu}{z^\lambda \Gamma(\mu)} \int_0^z t^{\lambda-1}\left(\log\frac{z}{t}\right)^{\mu-1} f(t)\,dt \\ &= z + \sum_{n=2}^\infty \left(\frac{\lambda+1}{\lambda+n}\right)^\mu a_n z^n \end{aligned} \tag{1.9}$$

$$(\lambda > -1;\ \mu > 0;\ f \in \mathcal{A}),$$

$$\begin{aligned} Q_\lambda^\mu f(z) &:= \binom{\lambda+\mu}{\lambda} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1}\left(1-\frac{t}{z}\right)^{\mu-1} f(t)\,dt \\ &= z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^\infty \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)} a_n z^n \end{aligned} \tag{1.10}$$

$$(\lambda > -1;\ \mu > 0;\ f \in \mathcal{A}),$$

and

$$\begin{aligned} J_\mu f(z) &:= \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)\,dt \\ &= z + \sum_{n=2}^\infty \left(\frac{\mu+1}{\mu+n}\right) a_n z^n \end{aligned} \tag{1.11}$$

$$(\mu > -1;\ f \in \mathcal{A}),$$

where Γ denotes the Gamma function, $f \in \mathcal{A}$ is assumed to be given by (1.1), and (in general)

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)\Gamma(\mu+1)} =: \binom{\lambda}{\lambda-\mu}. \tag{1.12}$$

The definitions (1.10) and (1.11) immediately yield the following: relationship,

$$J_\mu f(z) = Q_\mu^1 f(z) \qquad (\mu > -1). \tag{1.13}$$

The operators

$$P_1^\mu =: P^\mu, \qquad Q_\lambda^\mu, \qquad \text{and} \qquad J_\mu$$

were introduced and studied by Jung *et al.* [5] (and, subsequently, [6–14]). For

$$\mu = m \qquad (m \in \mathbb{N};\ \mathbb{N} := \{1,2,3,\dots\}),$$

these *special* integral operators were studied earlier by Bernardi [15,16]. Furthermore, just as already observed by (among others) Li and Srivastava [17, p. 135], the operator P^μ or P_1^μ is closely related to the *multiplier transformation* (or *fractional integral* and *fractional derivative*) investigated by Flett [18] and Kim *et al.* [19] (see also [20]).

The main object of this paper is to investigate the various inclusion relationships for each of the following subclasses of the *normalized* analytic function class $\mathcal{A}$, which are defined here by means of the Jung-Kim-Srivastava integral operator Q_λ^μ, given by (1.10).

DEFINITION 1. *In conjunction with (1.2) and (1.10),*

$$\mathcal{S}^*_{\lambda,\mu}(\alpha) := \{f : f \in \mathcal{A} \text{ and } Q_\lambda^\mu f \in \mathcal{S}^*(\alpha)\ \ (0 \leqq \alpha < 1)\}. \tag{1.14}$$

DEFINITION 2. *In conjunction with (1.3) and (1.10),*

$$\mathbb{K}_{\lambda,\mu}(\alpha) := \{f : f \in \mathcal{A} \text{ and } Q_\lambda^\mu f \in \mathbb{K}(\alpha)\ \ (0 \leqq \alpha < 1)\}. \tag{1.15}$$

DEFINITION 3. *In conjunction with (1.4) and (1.10),*

$$\mathcal{C}_{\lambda,\mu}(\beta,\alpha) := \{f : f \in \mathcal{A} \text{ and } Q_\lambda^\mu f \in \mathcal{C}(\beta,\alpha)\ \ (0 \leqq \alpha,\ \beta < 1)\}. \tag{1.16}$$

DEFINITION 4. *In conjunction with (1.5) and (1.10),*

$$\mathcal{C}^*_{\lambda,\mu}(\beta,\alpha) := \{f : f \in \mathcal{A} \text{ and } Q_\lambda^\mu f \in \mathcal{C}^*(\beta,\alpha)\ \ (0 \leqq \alpha,\ \beta < 1)\}. \tag{1.17}$$

In our investigation of the inclusion relationships involving the function classes given by Definitions 1–4 above, we shall make use of the following result (which is popularly known as the *Miller-Mocanu lemma*).

LEMMA. *(See [21].) Let $\Theta(u,v)$ be a complex-valued function, such that*

$$\Theta : \mathbb{D} \longrightarrow \mathbb{C} \qquad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

$\mathbb{C}$ being (as usual) the complex plane, and let

$$u = u_1 + iu_2 \qquad \text{and} \qquad v = v_1 + iv_2.$$

Suppose that the function $\Theta(u,v)$ satisfies each of the following conditions:

(i) *$\Theta(u,v)$ is continuous in $\mathbb{D}$,*
(ii) *$(1,0) \in \mathbb{D}$ and $\Re(\Theta(1,0)) > 0$,*
(iii) *$\Re(\Theta(iu_2,v_1)) \leqq 0$, for all $(iu_2,v_1) \in \mathbb{D}$, such that*

$$v_1 \leqq -\frac{1}{2}\left(1+u_2^2\right).$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{1.18}$$

be analytic (regular) in $\mathbb{U}$, *such that*

$$p(z) \not\equiv 1 \quad \text{and} \quad (p(z), zp'(z)) \in \mathbb{D} \qquad (z \in \mathbb{U}).$$

If

$$\Re(\Theta(p(z), zp'(z))) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

2. THE MAIN INCLUSION RELATIONSHIPS

We first prove the following result.

THEOREM 1. *Suppose that*

$$f \in \mathcal{A}, \quad \lambda > -1, \quad \mu > 0, \quad \text{and} \quad \lambda + \mu > -\alpha, \qquad (0 \leqq \alpha < 1).$$

Then

$$f \in \mathcal{S}^*_{\lambda,\mu}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{\lambda,\mu+1}(\alpha)$$

or, equivalently,

$$\mathcal{S}^*_{\lambda,\mu}(\alpha) \subset \mathcal{S}^*_{\lambda,\mu+1}(\alpha).$$

PROOF. Let $f \in \mathcal{S}^*_{\lambda,\mu}(\alpha)$ and assume that

$$\frac{z\left(Q_\lambda^{\mu+1} f(z)\right)'}{Q_\lambda^{\mu+1} f(z)} - \alpha = (1-\alpha)\,p(z), \tag{2.1}$$

where $p(z)$ is given by (1.18).

For the Jung-Kim-Srivastava integral operator Q_λ^μ defined by (1.10), it is known that

$$z\left(Q_\lambda^{\mu+1} f(z)\right)' = (\lambda+\mu+1)\,Q_\lambda^\mu f(z) - (\lambda+\mu)\,Q_\lambda^{\mu+1} f(z) \qquad (\lambda > -1;\ \mu > 0), \tag{2.2}$$

so that

$$\frac{(\lambda+\mu+1)\,Q_\lambda^\mu f(z)}{Q_\lambda^{\mu+1} f(z)} = \frac{z\left(Q_\lambda^{\mu+1} f(z)\right)'}{Q_\lambda^{\mu+1 f(z)}} + \lambda + \mu. \tag{2.3}$$

By combining (2.1) and (2.3), we obtain

$$\frac{Q_\lambda^\mu f(z)}{Q_\lambda^{\mu+1} f(z)} = \frac{1}{\lambda+\mu+1}\left[(1-\alpha)\,p(z) + \alpha + \lambda + \mu\right]. \tag{2.4}$$

Now, we logarithmically differentiate both sides of (2.4) with respect to z, and thus, we find that

$$\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu f(z)} = \frac{z\left(Q_\lambda^{\mu+1} f(z)\right)'}{Q_\lambda^{\mu+1} f(z)} + \frac{(1-\alpha)\,zp'(z)}{(1-\alpha)\,p(z) + \alpha + \lambda + \mu}, \tag{2.5}$$

which, in view of (2.1), yields

$$\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu f(z)} - \alpha = (1-\alpha)\,p(z) + \frac{(1-\alpha)\,zp'(z)}{(1-\alpha)\,p(z) + \alpha + \lambda + \mu}. \tag{2.6}$$

By taking

$$u = p(z) = u_1 + \imath u_2 \qquad \text{and} \qquad v = zp'(z) = v_1 + \imath v_2,$$

if we define the function $\Theta(u,v)$ by

$$\Theta(u,v) = (1-\alpha)u + \frac{(1-\alpha)v}{(1-\alpha)u + \alpha + \lambda + \mu}, \tag{2.7}$$

then, clearly, $\Theta(u,v)$ is continuous in

$$\mathbb{D} = \left(\mathbb{C} \Big\backslash \left\{\frac{\alpha+\lambda+\mu}{\alpha-1}\right\}\right) \times \mathbb{C}$$

and

$$(1,0) \in \mathbb{D}, \qquad \text{with } \Re(\Theta(1,0)) > 0.$$

Moreover, for all $(iu_2, v_1) \in \mathbb{D}$, such that

$$v_1 \leqq -\frac{1}{2}\left(1+u_2^2\right),$$

we have

$$\begin{aligned} \Re(\Theta(iu_2,v_1)) &= \Re\left(\frac{(1-\alpha)v_1}{(1-\alpha)iu_2+\alpha+\lambda+\mu}\right) \\ &= \frac{(1-\alpha)(\alpha+\lambda+\mu)v_1}{[(1-\alpha)u_2]^2+(\alpha+\lambda+\mu)^2} \\ &\leqq -\frac{(1-\alpha)(\alpha+\lambda+\mu)\left(1+u_2^2\right)}{2\left([(1-\alpha)u_2]^2+(\alpha+\lambda+\mu)^2\right)} \\ &< 0, \end{aligned} \tag{2.8}$$

which shows that $\Theta(u,v)$ satisfies the hypotheses of the Miller-Mocanu Lemma. Thus, by making use of (1.2), (2.1), and (2.6), we readily arrive at the inclusion relationship asserted by Theorem 1.

THEOREM 2. *Suppose that*

$$f \in \mathcal{A}, \quad \lambda > -1, \quad \mu > 0, \quad \text{and} \quad \lambda+\mu > -\alpha \qquad (0 \leqq \alpha < 1).$$

Then

$$f \in \mathbb{K}_{\lambda,\mu}(\alpha) \Longrightarrow f \in \mathbb{K}_{\lambda,\mu+1}(\alpha)$$

or, equivalently,

$$\mathbb{K}_{\lambda,\mu}(\alpha) \subset \mathbb{K}_{\lambda,\mu+1}(\alpha).$$

PROOF. Let $f \in \mathbb{K}_{\lambda,\mu}(\alpha)$. Then, by Definition 2,

$$Q_\lambda^\mu f \in \mathbb{K}(\alpha) \qquad (0 \leqq \alpha < 1;\ \lambda > -1;\ \mu > 0),$$

which, in view of (1.6), implies that

$$z\left(Q_\lambda^\mu f(z)\right)' \in \mathcal{S}^*(\alpha),$$

that is,

$$Q_\lambda^\mu\left(zf'(z)\right) \in \mathcal{S}^*(\alpha).$$

Thus, by Definition 1, we have

$$zf'(z) \in \mathcal{S}^*_{\lambda,\mu}(\alpha) \subset \mathcal{S}^*_{\lambda,\mu+1}(\alpha), \tag{2.9}$$

where we have also applied Theorem 1.

The inclusion relationship asserted by Theorem 2 would now follow easily from (2.9) in conjunction with (1.6), Definition 1, and Definition 2.

THEOREM 3. *Suppose that*

$$f \in \mathcal{A}, \quad \lambda > -1, \quad \mu > 0, \quad \text{and} \quad \lambda + \mu > -\alpha \qquad (0 \leqq \alpha < 1).$$

Then

$$f \in \mathcal{C}_{\lambda,\mu}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\lambda,\mu+1}(\beta,\alpha)$$

or, equivalently,

$$\mathcal{C}_{\lambda,\mu}(\beta,\alpha) \subset \mathcal{C}_{\lambda,\mu+1}(\beta,\alpha).$$

PROOF. Let $f \in \mathcal{C}_{\lambda,\mu}(\beta,\alpha)$. Then, by Definition 3, there exists a function $h \in \mathcal{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$, such that

$$\Re\left(\frac{z\left(Q_\lambda^\mu f(z)\right)'}{h(z)}\right) > \beta \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1). \tag{2.10}$$

By setting

$$h(z) = Q_\lambda^\mu g(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leqq \alpha < 1),$$

we find from Definition 1 and (2.10) that

$$g \in \mathcal{S}^*_{\lambda,\mu}(\alpha)$$

and

$$\Re\left(\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu g(z)}\right) > \beta \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1), \tag{2.11}$$

respectively.

We now let

$$\frac{z\left(Q_\lambda^{\mu+1} f(z)\right)'}{Q_\lambda^{\mu+1} g(z)} - \beta = (1-\beta)\,p(z), \tag{2.12}$$

where $p(z)$ is given by (1.18). Making use of the operator identity (2.2), we also have

$$\begin{aligned}
\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu g(z)} &= \frac{Q_\lambda^\mu\left(zf'(z)\right)}{Q_\lambda^\mu g(z)} \\
&= \frac{z\left(Q_\lambda^{\mu+1}\left(zf'(z)\right)\right)' + (\lambda+\mu)\,Q_\lambda^{\mu+1}\left(zf'(z)\right)}{z\left(Q_\lambda^{\mu+1} g(z)\right)' + (\lambda+\mu)\,Q_\lambda^{\mu+1} g(z)} \\
&= \left[\frac{z\left(Q_\lambda^{\mu+1}\left(zf'(z)\right)\right)'}{Q_\lambda^{\mu+1} g(z)} + (\lambda+\mu)\,\frac{Q_\lambda^{\mu+1}\left(zf'(z)\right)}{Q_\lambda^{\mu+1} g(z)}\right] \\
&\quad \cdot \left[\frac{z\left(Q_\lambda^{\mu+1} g(z)\right)'}{Q_\lambda^{\mu+1} g(z)} + \lambda + \mu\right]^{-1}.
\end{aligned} \tag{2.13}$$

By Theorem 1, we know that

$$g \in \mathcal{S}^*_{\lambda,\mu}(\alpha) \Longrightarrow g \in \mathcal{S}^*_{\lambda,\mu+1}(\alpha),$$

so that we can set

$$\frac{z\left(Q_\lambda^{\mu+1} g(z)\right)'}{Q_\lambda^{\mu+1} g(z)} = (1-\alpha)\,q(z) + \alpha, \tag{2 14}$$

where

$$\Re(q(z)) > 0 \qquad (z \in \mathbb{U}). \tag{2.15}$$

Upon substituting from (2.12) and (2.14) into (2.13), we have

$$\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu g(z)} = \frac{\left[z\left(Q_\lambda^{\mu+1}(zf'(z))\right)'\right]\cdot\left[Q_\lambda^{\mu+1}g(z)\right]^{-1} + (\lambda+\mu)\left[(1-\beta)p(z)+\beta\right]}{(1-\alpha)q(z)+\alpha+\lambda+\mu}. \tag{2.16}$$

By logarithmically differentiating both sides of (2.12) with respect to z, we also have

$$\frac{z\left(Q_\lambda^{\mu+1}(zf'(z))\right)'}{Q_\lambda^{\mu+1}g(z)} = (1-\beta)zp'(z) + \left[(1-\alpha)q(z)+\alpha\right]\left[(1-\beta)p(z)+\beta\right], \tag{2.17}$$

which, in conjunction with (2.16), yields

$$\frac{z\left(Q_\lambda^\mu f(z)\right)'}{Q_\lambda^\mu g(z)} - \beta = (1-\beta)p(z) + \frac{(1-\beta)zp'(z)}{(1-\alpha)q(z)+\alpha+\lambda+\mu} \tag{2.18}$$

The remainder of our proof of Theorem 3 is much akin to that of Theorem 1. Therefore, we choose to omit the analogous details involved.

Last, we prove an inclusion relationship involving the function class $\mathcal{C}^*_{\lambda,\mu}(\beta,\alpha)$ given by Definition 4.

THEOREM 4. *Suppose that*

$$f \in \mathcal{A}, \quad \lambda > -1, \quad \mu > 0, \quad \text{and} \quad \lambda+\mu > -\alpha, \qquad (0 \leqq \alpha < 1).$$

Then

$$f \in \mathcal{C}^*_{\lambda,\mu}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}^*_{\lambda,\mu+1}(\beta,\alpha)$$

or, equivalently,

$$\mathcal{C}^*_{\lambda,\mu}(\beta,\alpha) \subset \mathcal{C}^*_{\lambda,\mu+1}(\beta,\alpha).$$

PROOF. Just as we derived Theorem 2 as a consequence of Theorem 1 by means of the equivalence (1.6), we can prove Theorem 4 by appealing analogously to Theorem 3 and the equivalence (1.7).

3. REMARKS AND OBSERVATIONS

As already exhibited by (1.13), the special case of the integral operator Q_λ^μ when $\mu = 1$ is precisely the operator J_λ ($\lambda > -1$) defined by (1.11). Our main results (Theorems 1–4 above) can thus be applied with a view to deducing the following consequences.

COROLLARY 1. *Suppose that $f \in \mathcal{A}$ and $\lambda > -1$. Then*

$$f \in \mathcal{S}^*_{\lambda,1}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{\lambda,2}(\alpha).$$

Equivalently, if

$$J_\lambda f(z) \in \mathcal{S}^*(\alpha) \qquad (\lambda > -1;\ 0 \leqq \alpha < 1),$$

then

$$f \in \mathcal{S}^*_{\lambda,n}(\alpha) \quad (n \in \mathbb{N}\setminus\{1\}).$$

COROLLARY 2. *Suppose that $f \in \mathcal{A}$ and $\lambda > -1$. Then*

$$f \in \mathbb{K}_{\lambda,1}(\alpha) \Longrightarrow f \in \mathbb{K}_{\lambda,2}(\alpha).$$

Equivalently, if

$$J_\lambda f(z) \in \mathbb{K}(\alpha) \quad (\lambda > -1;\ 0 \leqq \alpha < 1),$$

then

$$f \in \mathbb{K}_{\lambda,n}(\alpha) \quad (n \in \mathbb{N}\setminus\{1\}).$$

COROLLARY 3. *Suppose that $f \in \mathcal{A}$ and $\lambda > -1$. Then*

$$f \in \mathcal{C}_{\lambda,1}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\lambda,2}(\beta,\alpha) \qquad (\lambda > -1;\ 0 \leqq \alpha,\ \beta < 1).$$

Equivalently, if

$$J_\lambda f(z) \in \mathcal{C}(\beta,\alpha) \qquad (\lambda > -1;\ 0 \leqq \alpha,\ \beta < 1),$$

then

$$f \in \mathcal{C}_{\lambda,n}(\beta,\alpha) \quad (n \in \mathbb{N}\setminus\{1\}).$$

COROLLARY 4. *Suppose that $f \in \mathcal{A}$ and $\lambda > -1$. Then*

$$f \in \mathcal{C}^*_{\lambda,1}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}^*_{\lambda,2}(\beta,\alpha) \qquad (\lambda > -1;\ 0 \leqq \alpha,\ \beta < 1).$$

Equivalently, if

$$J_\lambda f(z) \in \mathcal{C}^*(\beta,\alpha) \quad (\lambda > -1;\ 0 \leqq \alpha,\ \beta < 1),$$

then

$$f \in \mathcal{C}^*_{\lambda,n}(\beta,\alpha) \quad (n \in \mathbb{N}\setminus\{1\}).$$

Numerous other applications and consequences of our main results (Theorems 1 to 4) can indeed be derived similarly.

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